EXO 1 TD

$N = 5$, $n = 2$

$y = \{8, 3, 11, 4, 7\}$

1) Calcul de $\bar{y}$ et S_y^2

- $\bar{y} = \frac{1}{N} \sum_{i \in U} y_i$

$= \frac{1}{5}(8+3+11+4+7)$

$\underline{\bar{y} = 6{,}6}$

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$= \frac{1}{5}(8+3+11+4+7)$

$\bar{y} = 6,6$

• $S_y^2 = \frac{1}{N-1} \sum_{k \in U} (y_k - \bar{y})^2$

$= \frac{1}{4}\left[(8-6,6)^2 + (3-6,6)^2 + (11-6,6)^2 + (4-6,6)^2 + (7-6,6)^2\right]$

$S_y^2 = 10,3$

2) • On sait que le nombre d'échantillon est donné par C_N^n.

donc on a : $C_5^2 = 10$ échantillons.

	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9	S_{10}
	$(8,3)$	$(8,11)$	$(8,4)$	$(8,7)$	$(3,11)$	$(3,4)$	$(3,7)$	$(11,4)$	$(11,7)$	$(4,7)$
$\hat{y}_i$	5,5	9,5	6	7,5	7	3,5	5	7,5	9	5,5

4) Vérifions que $\hat{y}$ estime sans biais $\bar{Y}$.

Il suffit de montrer que

$$E(\hat{y}) = \bar{Y}$$

$$E(\hat{y}) = \frac{1}{N}\sum_{i=1}^{10} \hat{y}_i$$

$$E(\hat{y}) = \frac{1}{10}\left(5{,}5 + 9{,}5 + 6 + 7{,}5 + 7 + 3{,}5 + 5 + 7{,}5 + 9 + 5{,}5\right)$$

$$= 6{,}6$$

$$= \bar{y}$$

donc $\hat{y}$ est un estimateur sans biais de $\bar{y}$.

5) Variance de $\hat{y}$.

$$Var(\hat{y}) = \frac{1}{n^*}\sum\left(\hat{y}_\alpha - \bar{y}\right)^2 \quad \text{avec } n^* = 10$$

$$= \frac{1}{10}\Big[(5{,}5-6{,}6)^2 + (9{,}5-6{,}6)^2 + (6-6{,}6)^2 + (7{,}5-6{,}6)^2$$
$$+ (7-6{,}6)^2 + (3{,}5-6{,}6)^2 + (5-6{,}6)^2 + (7{,}5-6{,}6)^2$$
$$+ (9-6{,}6)^2 + (5{,}5-6{,}6)^2\Big]$$

$$\underline{\underline{Var(\hat{\bar{y}}) = 3{,}09}}$$

c) Vérifions que $Var(\hat{\bar{y}})$

Coïncide avec la formule théorique

$$\boxed{Var(\hat{\bar{y}}) = \left(1-\frac{n}{N}\right)\frac{S_y^2}{n}}$$

$$Var(\hat{\bar{y}}) = \left(1-\frac{2}{5}\right) \times \frac{10{,}3}{2}$$

$$Var(\hat{\bar{y}}) = 3{,}09$$

7) Calculons $\widehat{Var}\left(\frac{\hat{y}}{g}\right)$

on sait que :

$$\widehat{Var}\left(\frac{\hat{y}}{g_i}\right) = \left(1-\frac{n}{N}\right)\frac{\hat{S}_{i}^{2}}{n}$$

- $\hat{S}_1^2 = \frac{1}{2-1}\left[(8-5,5)^2 + (3-5,5)^2\right] = 12,5$.

$$\widehat{Var}\left(\frac{\hat{y}}{g_1}\right) = \left(1-\frac{2}{5}\right)\frac{12,5}{2} = 3,75$$

- $\hat{S}_2^2 = \left[(8-9,5)^2 + (11-9,5)^2\right] = 4,5$

$$\widehat{Var}\left(\frac{\hat{y}}{g_2}\right) = \left(1-\frac{2}{5}\right)\frac{4,5}{2} = 1,35$$

thérique.

	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9	S_{10}
	(8,3)	(8,11)	(8,4)	(8,7)	(3,11)	(3,4)	(3,7)	(11,4)	(11,7)	(4,7)
$\hat{\theta}_i$	5,5	9,5	6	7,5	7	3,5	5	7,5	9	5,5
$\hat{Var}(\hat{\theta}_i)$	3,75	1,35	2,4	0,15	9,6					

$$\hat{S}_3^2 = \left[(8-6)^2+(4-6)^2\right] = 8$$

$$\hat{Var}(\hat{\theta}_3) = \left(1-\frac{2}{5}\right)\frac{8}{2} = 2,4$$

$$\hat{S}_4^2 = \left[(8-7,5)^2+(7-7,5)^2\right] = 0,5$$

$$\hat{Var}(\hat{\theta}_4) = \left(1-\frac{2}{5}\right)\frac{0,5}{2} = 0,15$$

$$\hat{S}_5^2 = \left[(3-7)^2 + (11-7)^2\right] = 32$$

$$\widehat{Var}(\hat{\bar{y}}_{s_5}) = \left(1-\frac{2}{5}\right)\frac{32}{2} = 9{,}6$$

$$\hat{S}_6^2 = \left[(3-3{,}5)^2 + (4-3{,}5)^2\right] = 0{,}5$$

$$\widehat{Var}(\hat{\bar{y}}_{s_6}) = \left(1-\frac{2}{5}\right)\frac{0{,}5}{2} = 0{,}15$$

$$\hat{S}_7^2 = \left[(3-5)^2 + (7-5)^2\right] = 8$$

$$\widehat{Var}(\hat{\bar{y}}_{s_7}) = \left(1-\frac{2}{5}\right)\frac{8}{2} = 2{,}4$$

$$\hat{S}_8^2 = \left[(11-7{,}5)^2 + (4-7{,}5)^2\right] = 24{,}5$$

$$\widehat{Var}(\hat{\bar{y}}_{s_8}) = \left(1-\frac{2}{5}\right)\frac{24{,}5}{2} = 7{,}35$$

	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9	S_{10}
	(8,3)	(8,11)	(8,4)	(8,7)	(3,11)	(3,4)	(3,7)	(11,4)	(11,7)	(4,7)
$\hat{\theta}_i$	5,5	9,5	6	7,5	7	3,5	5	7,5	9	5,5
$\widehat{Var}(\hat{\theta}_i)$	3,75	1,35	2,4	0,15	3,6	0,15	3,4	1,35	2,4	1,35

$$\hat{S}_9^2 = [(11-9)^2 + (7-9)^2] = 8$$

$$\widehat{Var}(\bar{a}_9) = \left(1-\frac{2}{5}\right)\frac{8}{2} = 2{,}4$$

$$\hat{S}_{10}^2 = [(4-5{,}5)^2 + (7-5{,}5)^2] = 4{,}5$$

$$\widehat{Var}(\bar{a}_{10}) = \left(1-\frac{2}{5}\right)\frac{4{,}5}{2} = 1{,}35$$

8) $E(\widehat{Var}(\hat{\theta})) = \frac{1}{10} \sum_i [\widehat{Var}(\hat{\theta}_i)]$

$= \frac{1}{10} [3{,}75 + 1{,}35 + 2{,}4 + 0{,}15 + 9{,}6 + 0{,}15 + 2{,}35 + 7{,}35 + 2{,}4 + 1{,}35]$

$E(\widehat{Var}(\hat{\theta})) = 3{,}09$

dnč. $E(\widehat{Var}(\hat{\theta})) = Var(\hat{\theta})$

EXO3 TD

On donne : $U=\{1,2,3\}$

$n=2$. $P(1,2)=\frac{1}{2}$; $P(1,3)=\frac{1}{4}$

$P(2,3)=\frac{1}{4}$

1)

2) $P(k\in S)=\sum_{S\ni k} P(S)$

- $P(1\in S)=P(1,2)+P(1,3)=\frac{3}{4}$
- $P(2\in S)=\frac{1}{2}+\frac{1}{4}=\frac{3}{4}$.

$P(3 \in S) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

3) Calculons $\hat{y}$ pour chaque échantillon :

$\hat{y}_1 = 1,5$, $\hat{y}_2 = 2$, $\hat{y}_3 = 2,5$

4) Vérifions que $E(\hat{y}) \neq \bar{y}$

$E(\hat{y}) = \left[\frac{1}{2} \times (1,5) + \frac{1}{4}(2) + \frac{1}{4}(2,5)\right] = 1,875$

or $\bar{y} = \frac{1+2+3}{3} = 2$ donc $\hat{y}$ est biaisé.

Exo 4

$U = \{98,7 \quad 102,6 \quad 108,5 \quad 120,3\}$

a) determiner la moyenne et la variance

$\bar{y} = \frac{1}{N} \sum y_k$

$\bar{y} = \frac{1}{4} (98,7 + 102,6 + 108,5 + 120,3)$

$\bar{y} = 107,525$

$\sigma^2 = \frac{1}{N} \sum (y_k - \bar{y})^2$

$\sigma_y^2 = \frac{1}{4} (98,7 - 107,525)^2 + (102,6 - 107,525)^2 + (108,5 - 10,525)^2 +$

$(120,3 - 10,525)^2$

$\sigma_y^2 = 66,57$

b) i) Nombre d'echantillon de taille 3

$C_4^3 = 4$

ii) Cites

échantillon		
	(98,7 , 102,6 , 108.5	(98,7 . 102,6 . 120,3
	98,7 ; 108,5 ; 120,3	(102,6; 108,5 ; 120,3

c)

Echantillon	Moyenne	Variance $\hat{\sigma}^2_y$
98,7 . 102,6 . 108,5	103,27	16,22
98,7 . 102,6 . 120,3	107,2	88,34
98,7 . 108,5 . 120,3	109,16	77,98
102,6 . 108,5 . 120,3	110,46	54,14

d) Donner la distribution de $\hat{\bar{y}}$

$\hat{\bar{y}}$	103,27	107,2	109,16	110,46
Probabilité	1/4	1/4	1/4	1/4

d) Donner la distribution de $\hat{y}$

$\hat{y}$	108,27	107,2	107,16	110,46
Probabilité	1/4	1/4	1/4	1/4

$E(\hat{\bar{y}}) = \frac{1}{4} \times (108,27) + \frac{1}{4}(107,2) + \frac{1}{4}(107,16) + \frac{1}{4}(110,40)$

$E(\hat{\bar{y}}) = 107,5255$

- calcul du Biais et EQM

$B = E(\hat{\bar{y}}) - \bar{y} = 107,5255 - 107,525 = 0,0005$

$EQM = (B)^2 + V(\hat{\bar{y}}) = (0,0005)^2 + \cancel{66,57}\ 15,7 = 66,57$

e) donnons la distribution de $\hat{\bar{y}}$

échantillon mâle : $(102,6 \cdot 108,5 \cdot 126,3$ $P = \frac{2}{5}$

$\hat{\bar{y}}$	103,27	107,2	109,16	110,46
Probabilité	1/5	1/5	1/5	2/5

- Calculons $E(\hat{\bar{y}})$

$E(\hat{\bar{y}}) = \frac{1}{5}(103,27) + \frac{1}{5}(107,2) + \frac{1}{5}(109,16) + \frac{2}{5}(110,46)$

$E(\hat{\bar{y}}) = 108,11$

- Calcul du biais

$B = E(\hat{\bar{y}}) - \bar{Y} = 108,11 - 107,525 = 0,585$

- Calcul de EQM

$EQM = (B)^2 + \hat{Var}(\hat{\bar{y}})$

$\hat{Var}(\hat{\bar{y}}) = \sum P_i (\hat{\bar{y}}_i)^2 - E($

$\hat{Var}(\hat{\bar{y}}) = \frac{1}{5}(103,27)^2 + \frac{1}{5}(107,2)^2 + \frac{1}{5}(109,6)^2 + \frac{2}{5}(110,46)^2 - (108,11)^2$

$= 7,28024$

donc $EQM = (0,585)^2 + 7,28024 = 7,622465$

<u>Exo 6</u>

$\mathrm{ar}(\hat{y}) = \frac{1}{5}(103{,}27)^2 + \frac{1}{5}(107{,}2)^2 + \frac{1}{5}(109{,}6)^2 + \frac{2}{5}(110{,}46)^2 - (109{,}4$

$= 7{,}28024$

donc $EQM = (0{,}585)^2 + 7{,}28024 = 7{,}622465$

Exo 6

$Z_k \sim U(0,1)$

1) probabilité d'un ordre particulier

$P = \frac{1}{N!}$

2) Probabilité que l'ordre commence par n unité (ordonné)

$$P = \frac{1 \times (N-n)}{N!}$$

3) Deduire la probabilité que

$$P = \frac{n!(N-n)}{N!} = \frac{1}{C_N^n}$$

Exercice 7

Exercice 7

1) Donnons la valeur de $\hat{\mu} = \bar{x}$

$\hat{\mu} = \bar{x} = \frac{1}{n} \sum x_i$

$= \frac{1}{100} \times 2907$

$\hat{\mu} = 29,07$

2) Donnons IC 95%

$IC = \left[\hat{\mu} \pm Z_{1-\frac{\alpha}{2}} \sqrt{(1-f)\frac{s^2}{n}}\right.$

$* \; f = \frac{n}{N} = \frac{100}{2010} =$

$\sigma^2 = \frac{1}{n} \sum x_i^2 - \bar{x}^2$ et $s^2 = \frac{n\sigma^2}{n-1}$

$= \frac{1}{100}(154593) - (29,07)^2$

$= 700,8551$

$$S^2 = \frac{100}{99} \cdot 700,8651 = \underline{707,9445}$$

$\alpha = 5\%$ $\quad 1-\frac{\alpha}{2} = 0,975, \quad z_{1-\frac{\alpha}{2}} = 1,96$

$$IC = \left[29,07 - 1,96\sqrt{\left(1-\frac{100}{2010}\right)\frac{707,9445}{100}}\right]$$

$$IC\ [23,99\ ;\ 34,15]$$

Exercice 12

EXO 12

	N_h	M_h	σ_{he}^2
Mâles	60	6	4
femelle	40	4	2,25

1) $$\sigma^2 = \frac{1}{N}\sum_{h=1}^{H} N_h \sigma_h^2 + \frac{1}{N}\sum_{1}^{H} N_h (M_h - M)^2$$

* Cherchons M et σ_h^2

$$M = \frac{1}{N}\sum N_h M_h = \frac{1}{100}(60 \times 6 + 40 \times 4)$$

$$\boxed{M = 5,2}$$

$$\sigma_h^2 = \frac{N_h - 1}{N_h}\sigma_{he}^2$$

$\sigma_1^2 =$

$\sigma_2^2 =$

σ^2 intra

σ^2 inte

$$\sum N_h (M_h - M)^2$$

$$\sigma_h^2$$

$$\frac{1}{100}(60 \times 6 + 40 \times 4)$$

$$\sigma_1^2 = \frac{59}{60} \times 4 = 3{,}933$$

$$\sigma_2^2 = \frac{39}{40} \times 2{,}25 = 2{,}19$$

$$\sigma^2_{intra} = \frac{1}{100}\left(60 \times 3{,}933 + 40 \times 2{,}19\right) = 3{,}2358$$

$$\sigma^2_{inter} = \frac{1}{100}\left(60 \times (6 - 5{,}2)^2 + 40(4 - 5{,}2)^2\right) = 0{,}96$$

$$\sigma^2 = \sigma^2_{intra} + \sigma^2_{inter}$$

$$= 3{,}2358 + 0{,}96$$

$$\boxed{\sigma^2 = 4{,}1955}$$

2) $V(\hat{t}_y) = N^2 \, V(\hat{\mu})$

avec $V(\hat{\mu}) = \left(1 - \frac{n}{N}\right)\frac{S^2}{n}$

on sait $S^2 = \frac{N}{N-1}\sigma^2$

$S^2 = \frac{100}{99} \times 4,1955$

$V(\hat{\mu}) = 0,3814$

$V(\hat{t}_y) = (100)^2 \times 0,3814$

$\boxed{V(\hat{t}_y) = 3814,09}$

3) $n = 10$

$V(\hat{t}_y) = \frac{N-n}{n}\sum N_h \sigma^2_{h,e}$

$V(\hat{t}_y) = \frac{100-10}{10}(60 \times 4 + 40 \times 2,25)$

$\boxed{V(\hat{t}_y) = 2970}$

4) $n = 10$

* Cherchons la répartition

$n_h = n \frac{N_h \sigma_{h,e}}{\sum N_h \sigma_{h,e}}$

$\sum N_h \sigma_{h,e} = 60 \times \sqrt{4} + 40 \times \sqrt{2,25}$
$= 180$

$n_1 = 10 \frac{60 \times 2}{180} = 6,66 \simeq 7$

$n_2 = 10 \frac{40 \times \sqrt{2,25}}{180} = 3,33 \simeq 3$

* $V(\hat{t}_y) = \sum_{h=1}^{H} N_h \frac{N_h - n_h}{n_h} \sigma^2_{h,e}$

$V(\hat{t}_y) = 60 \frac{60-7}{7} \times 4 + 40 \frac{40-3}{3} \times 2,25$

$\boxed{V(\hat{t}_y) = }$

Ex

Mâles

femelles

1)

$\sigma^2 =$

* Ch

$M =$

$\boxed{M =}$

σ_h